Spin of Dirac's relativistic membrane

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The Hamiltonian of Dirac's relativistic membrane is linearized by means of a gauge transformation. It is pointed out that the membrane spin is $\pm\hbar/2$. Furthermore, in a quantum relativistic framework neutrinolike particles are obtained when the membrane charge vanishes. [S1063-651X(99)02805-6]

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I. INTRODUCTION

The history of membrane theories can be traced back decades, starting from the papers by Yukawa [1], Dirac [2], and later by Collins and Tucker [3]. Recently the study of membranes has enjoyed great interest, due to their formal similarity to string theories and also because of some attractive characteristics unique to membranes, such as the idea that their spectrum of excitations is richer than those afforded by strings, and moreover, that a superstring theory is merely the limiting case of a compactified supermembrane. Notable developments in the study of extended models for particles have been considered within the context proposed by Dirac, where a relativistic membrane should describe an electrically charged particle. Much attention has been focused on theories of two-dimensional immersions (relativistic strings) in a four-dimensional space-time. Also, it is expected that their superspace extensions may provide a basic theory unifying gauge fields with matter. If we adopt a membrane model for particles where certain interactions with an external field are postulated, then it is possible to find a classical equilibrium configuration about which we can attempt a quantum treatment.

Relativistic membranes have been studied on a classical level by several authors [4], where some alternative forms of classical membrane action as well as supersymmetric extensions have been considered [5]. Dynamical properties of a charged membrane immersed in a Minkowski space-time have been studied by Onder and Tucker [6], taking into account radial modes of oscillations. Later, the same authors developed a semiclassical approach with a variational estimate of the energy levels of a particular type of a charged membrane [7]. However, the most important membrane approach is the Dirac framework, where the electron is a charged conducting closed surface, with Maxwell's free equation holding outside it and where Dirac has imposed a surface tension to prevent this surface from flying apart. In this case any stabilizing force that would counteract the membrane tension would act as a pressure. This is merely the unconstrained ''soap-bubble'' problem in a new guise, whose experimental results produce a spherical configuration. The classical equation of motion of such a charged spherically symmetric membrane describes an oscillatory radial motion about its center of symmetry, contracting from a maximum radius down to a point [8] and then expanding again. From the quantum-mechanical point of view it costs infinite energy to reduce the membrane to a point, and so we might expect a stable configuration about its equilibrium position, where, according to the uncertainty principle $\Delta r \Delta p \approx \hbar$ the membrane tension is balanced by the electrostatic force.

The Dirac model for the electron [2] is the most natural concept that causes its total energy to be finite due to the Coulomb field, and which should allow one to investigate whether the muon could be an electron in an excited state. However, in such a framework the spin and the quantum energy levels are still open problems. The aim of the present paper is twofold: first, we propose a method of linearization for the Hamiltonian of Dirac's relativistic membrane; second, we study its spin and derive its quantum energy by means of Dirac's equation, at the limiting case when the membrane charge vanishes.

II. REVIEW ON THE CLASSICAL APPROACH

The Dirac formalism of a charged relativistic membrane is developed in a Minkowski space-time $\mathcal{M}$, where the membrane is a (2+1)-dimensional manifold $\mathcal{T}$ imbedded in $\mathcal{M}$. The action integral for the electromagnetic field outside the membrane is, in spherical coordinates,

$$I = -\frac{1}{16\pi}\int_{\mathcal{M}} \sqrt{g} F_{\mu\nu} F^{\mu\nu} dt\, dR\, d\theta\, d\phi, \qquad (2.1)$$

where $g = |\det g_{\mu\nu}|$ is concerning the space-time metric (Greek indices have the range 0, 1, 2, 3 and we adopt a system of units where $c=1$ and $\hbar = 2.6054 \times 10^{-70} m^2$). The condition $\delta I = 0$ leads to Maxwell's free equation $\partial_\mu F^{\mu\nu} = 0$. Otherwise, the whole action integral for the membrane consists of an action integral for a free membrane [3] to which we add the action due to the electromagnetic field on the surface of the manifold $\mathcal{T}$:

$$S = -\frac{\Lambda}{4\pi}\int_{\mathcal{T}} \sqrt{\bar{g}}\, dt\, d\theta\, d\phi + \frac{1}{8\pi}\int_{\mathcal{T}} \sqrt{g} F_{\mu\nu} F^{\mu\nu} dt\, dR\, d\theta\, d\phi, \qquad (2.2)$$

where $\bar{g}$ is the determinant of the induced metric on the surface of $\mathcal{T}$ and Λ is a constant. Since the membrane dy-

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namics involves the components of the induced metric, then from the practical viewpoint we should provide a global parametrization. However, this is not in general possible with a single coordinate system. Such a problem is intimately connected with the choice of a proper gauge, which should not constrain the motion of the membrane. In the case of a spherically symmetric membrane we take for the space-time metric

$$g_{\mu\nu}=\mathrm{diag}[1,-1,-R^2,-R^2\sin^2\theta]. \tag{2.3}$$

However, if we consider the transformation of the radial coordinate in the form $X^1=R-r$, we obtain the induced metric on the surface $\mathcal{T}$ by deleting column 1 and row 1 in the space-time matrix given above [3]. This yields

$$[\overline{g}]=\begin{pmatrix} 1-\dot{r}^2 & 0 & 0 \\ 0 & -(X^1+r)^2 & 0 \\ 0 & 0 & -(X^1+r)^2\sin^2\theta \end{pmatrix} \tag{2.4}$$

and then, in terms of the coordinates t, θ, and ϕ, the surface of the membrane has the equation $X^1=0$ or $R=r(t)$.

The effective Lagrangian of the membrane can be derived in the coordinate system where $X^1=0$, if we take into account Eqs. (2.2), (2.3), and (2.4). Since just outside the membrane the field is merely Coulombian, i.e., $F_{\mu\nu}F^{\mu\nu}=e^2/r^4$, where e is the charge of the membrane, then we get

$$L(r,\dot{r})=-\Lambda r^2\sqrt{1-\dot{r}^2}-\frac{e^2}{2r}. \tag{2.5}$$

By means of the Euler-Lagrange equation the above Lagrangian leads straightforwardly to the equation of motion

$$\frac{d}{dt}\left[\frac{\Lambda\dot{r}r^2}{\sqrt{1-\dot{r}^2}}\right]+2\Lambda r\sqrt{1-\dot{r}^2}=\frac{e^2}{2r^2}. \tag{2.6}$$

If we impose that at the equilibrium position $r=a$ we must have $\ddot{r}(a)=\dot{r}(a)=0$, we obtain $\Lambda=e^2/4a^3$, and then Eq. (2.6) becomes

$$\ddot{r}+\frac{2}{r}(1-\dot{r}^2)=\frac{2a^3}{r^4}(1-\dot{r}^2)^{3/2}. \tag{2.7}$$

Notice that this equation has regular cycles in the phase plane $\dot{r}$ versus r, about the equilibrium configuration for all initial conditions. Searching for a solution to the above equation, we consider a change of the integration variable in the form $f(r)=1-\dot{r}^2$, and then we are led to a Bernoulli equation for $f(r)$, whose solution yields the exact solution

$$\dot{r}=\sqrt{1-\frac{r^6}{(kr-2a^3)^2}}, \tag{2.8}$$

where the constant k can assume the real values $3a^2$ and a^2, since these values lead to $\dot{r}(a)=0$. Moreover, the corresponding radial momentum is

$$p(r,\dot{r})=\frac{\partial L}{\partial\dot{r}}=\frac{\mu\dot{r}}{\sqrt{1-\dot{r}^2}}, \tag{2.9}$$

where $\mu=e^2r^2/(4a^3)$ and the other components of the canonical momentum vanish identically. Thus, in the present approach the membrane has only radial modes of oscillations, and it behaves as a spherical oscillating bubble. Finally, taking into account Eqs. (2.5) and (2.9) and since the classical Hamiltonian is given by $H=p\dot{r}-L$, we obtain

$$H=\sqrt{p^2+\mu^2(r)}+V(r), \tag{2.10}$$

which is the energy-momentum relation of the membrane, where $V(r)=e^2/(2r)$. It is important to remark that the relativistic Lagrangian (2.5) is similar to the Lagrangian of a particle under the influence of an external potential and that Eq. (2.10) is similar to its Hamiltonian, where $\mu(r)$ replaces the rest mass of the particle and **p** replaces its linear momentum of translational motion. Moreover, by direct substitution of Eq. (2.8) into Eq. (2.9), we conclude from Eq. (2.10) that at $r=a$ we have $H=ke^2/4a^3$. With $k=3a^2$ we obtain $H=3e^2/(4a)$ and with $k=a^2$ we get $H=e^2/(4a)$. Hence, both values of k define two different values for the energy of the membrane at $r=a$, since $H(a)=\pm\mu(a)+V(a)$. In particular, if the charge of the membrane is the electron charge $e=1.38\times10^{-36}$ m, and if its equilibrium radius is the electron classical radius $a=2.11\times10^{-15}$ m, then we get with $k=3a^2$, $H=6.76\times10^{-58}$ $m=0.51$ MeV, which is the electron rest mass.

With the Bohr-Sommerfeld quantization method, Dirac [2] has shown that the energy of the first excited state of the membrane is about a quarter of the muon mass. Moreover, with a model of a charged membrane different from the Dirac model, Onder and Tucker [6] have derived a reasonable value for the muon mass at the first level of excitation, in the context of a three-dimensional oscillator. Because both procedures above are semiclassical, their results are not trustworthy. With the purpose of finding a relativistic wave equation in the form $i\hbar\partial\Psi(r,t)/\partial t=H\Psi(r,t)$, we notice that the kinetic term T of the Hamiltonian (2.10) is the positive square root of $p^2+\mu^2$. Therefore, if p is replaced by $-i\hbar\partial/\partial r$ and T is substituted into the above wave equation, the resulting wave equation is unsymmetrical with respect to space and time derivatives, and hence it is not relativistic. Thus, we have to modify the Hamiltonian (2.10) in such a way as to make it linear in space derivatives.

III. LINEARIZATION OF THE SQUARE ROOT OPERATOR IN SPACE DERIVATIVES

Let us extend the standard Dirac method of linearization, starting from Eq. (2.10) written in the form $H=T+V(r)$, where $T=\sqrt{p^2+\mu^2}$. In order to linearize this square root we perform a gauge transformation of the momentum **p** and insert an additional term in T. Since the characteristics of the membrane should not depend on the choice of the matrices employed in the linearization process, then let us consider, for instance, the matrices

$$\boldsymbol{\alpha}=\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & \mathbf{0}\end{pmatrix},\quad \boldsymbol{\beta}=\begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{0} & -\mathbf{I}\end{pmatrix},\quad \boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\sigma}\end{pmatrix}, \tag{3.1}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are Dirac matrices, $\boldsymbol{\sigma}$ are the Pauli matrices

$$\boldsymbol{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \boldsymbol{\sigma}_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \boldsymbol{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{3.2}$$

and $\mathbf{I}$ is the identity matrix. We remind the reader that the matrices of Eq. (3.1) satisfy the relations

$$\boldsymbol{\alpha}_j^2=\boldsymbol{\beta}^2=\boldsymbol{\Sigma}_j^2=\mathbf{I},\quad \{\boldsymbol{\alpha}_j,\boldsymbol{\beta}\}=\mathbf{0},$$

$$\{\boldsymbol{\alpha}_j,\boldsymbol{\alpha}_\kappa\}=2\delta_{jk}\mathbf{I},$$

$$[\boldsymbol{\Sigma}_j,\boldsymbol{\alpha}_\kappa]=2i\in_{jkl}\boldsymbol{\alpha}_l,\quad [\boldsymbol{\Sigma}_l,\boldsymbol{\beta}]=\mathbf{0}, \tag{3.3}$$

where $j,k,l=1,2,3$ are in cyclic order. We then choose T in the form

$$T=\boldsymbol{\alpha}\cdot[\mathbf{p}-i\mathbf{q}]+i(\boldsymbol{\alpha}\cdot\mathbf{f})\boldsymbol{\beta}K+\boldsymbol{\beta}\mu, \tag{3.4}$$

where $\mathbf{q}$ and $\mathbf{f}$ are real vectors and K is an arbitrary operator. Since $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{f}$ commute with $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, then let us assume that the operator K commutes with $\mathbf{p}-i\mathbf{q}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$ as well. However, the Hermitian conjugate of the operator T is

$$T^\dagger=\boldsymbol{\alpha}\cdot[\mathbf{p}+i\mathbf{q}]-iK^\dagger\boldsymbol{\beta}(\boldsymbol{\alpha}\cdot\mathbf{f})+\boldsymbol{\beta}\mu, \tag{3.5}$$

where $K^\dagger$ is assumed to commute also with $\mathbf{p}-i\mathbf{q}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$. Since the membrane has only radial modes of oscillations, then let us take spherical coordinates with origin at its center of symmetry, and let us operate with the square of T on an arbitrary function $\Phi(r,\theta,\phi)$:

$$\begin{aligned}\mathbf{I}T^\dagger T\Phi=(&p^2+\mu^2+q^2-i[p,\ q]+ipf\boldsymbol{\beta}K^\dagger-iK^\dagger\boldsymbol{\beta}fp\\&+\boldsymbol{\alpha}\boldsymbol{\beta}[p,\ \mu]-qf\boldsymbol{\beta}K-K^\dagger\boldsymbol{\beta}fq+2i\boldsymbol{\alpha}\boldsymbol{\beta}q\mu\\&+K^\dagger Kf^2+iK^\dagger\boldsymbol{\alpha}f\mu-i\boldsymbol{\alpha}\mu fK)\Phi.\end{aligned} \tag{3.6}$$

By comparing both sides of the above equation, we see that the first requirement to obtain the desired linearization is that K must be a linear operator. Moreover, if κ and κ^* are the respective eigenvalues of the operators K and $K^\dagger$, then we also have to assume that K should be a Hermitian operator. Hence, replacing p by $-i\hbar\boldsymbol{\nabla}$ we get from Eq. (3.6)

$$\begin{aligned}\mathbf{I}T^2=&[p^2+\mu^2+q^2+\kappa^2f^2-\hbar(\boldsymbol{\nabla}\cdot\mathbf{q})]\mathbf{I}\\&+[\hbar(\boldsymbol{\nabla}\cdot\mathbf{f})-2qf]\kappa\boldsymbol{\beta}-i[\hbar(\boldsymbol{\nabla}\mu)-2q\mu]\boldsymbol{\alpha}\boldsymbol{\beta}.\end{aligned} \tag{3.7}$$

From the third term of Eq. (3.7) we obtain $|\mathbf{q}|=\hbar/r$. Hence, the desired linearization leads to the following system of independent equations:

$$q^2+\kappa^2f^2-\hbar(\boldsymbol{\nabla}\cdot\mathbf{q})\pm\kappa\hbar(\boldsymbol{\nabla}\cdot\mathbf{f})\mp2\kappa qf=0 \tag{3.8}$$

whose respective solutions are $|\mathbf{f}|=\hbar/r$ and $\kappa=\pm1$. Therefore, considering Eq. (3.4) and the above results, the linearized form of the Hamiltonian (2.10) is in spherical coordinates

$$H=\boldsymbol{\alpha}\cdot\mathbf{P}+i\frac{\hbar}{r}\boldsymbol{\alpha}\boldsymbol{\beta}K+\boldsymbol{\beta}\mu+\mathbf{I}V, \tag{3.9}$$

where $P=p-i\hbar/r$ and $p=-i\hbar\partial/\partial r$. The physical interpretation of the operator K will be given at the end of Sec. V.

For the sake of consistency it is important to remark that, according to Eq. (3.7), if we had on the other hand $\mu=m_0$ (constant), we would obtain $\mathbf{q}=\mathbf{f}=\mathbf{0}$, which turns Eq. (3.4) into $T=\boldsymbol{\alpha}\cdot\mathbf{p}+\boldsymbol{\beta}m_0$. We recall that this is the kinetic term of the relativistic Hamiltonian operator of a particle with rest mass m_0, if $\mathbf{p}$ is its linear momentum of translational motion. Thus, in this sense, the present method generalizes the process of linearization proposed by Dirac for a relativistic Hamiltonian operator.

IV. LARGE AND SMALL COMPONENTS OF THE WAVE FUNCTION

We can write the wave equation of the membrane as a Schrödinger equation in a matrix form $i\hbar\partial\boldsymbol{\Psi}/\partial t=H\boldsymbol{\Psi}$, if we take into account the Hamiltonian operator (3.9) written as

$$H=\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}+\boldsymbol{\beta}\mu+\mathbf{I}V, \tag{4.1}$$

where $\boldsymbol{\Pi}=\mathbf{P}+i\mathbf{q}\boldsymbol{\beta}K$. However, for stationary states, the time dependence of the wave function can be split off according to $\boldsymbol{\Psi}(\mathbf{r},t)=\boldsymbol{\Psi}(\mathbf{r})\exp[-i\mathcal{E}t/\hbar]$, where $\mathcal{E}(r)$ defines the behavior of the time dependence of the complete wave function. Recall that $\mathcal{E}(r)$ is such that the matrix equation

$$H\boldsymbol{\Psi}(\mathbf{r})=\mathcal{E}\boldsymbol{\Psi}(\mathbf{r}) \tag{4.2}$$

must be satisfied. Since we can express the wave function $\boldsymbol{\Psi}(\mathbf{r})$ in terms of two-component functions $\boldsymbol{\varphi}=\binom{\Psi_1}{\Psi_2}$ and $\boldsymbol{\chi}=\binom{\Psi_3}{\Psi_4}$ through the form $\boldsymbol{\Psi}(\mathbf{r})=\binom{\varphi}{\chi}$ then, owing to the definitions of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ given in Eq. (3.1), we obtain from Eq. (4.2) the system of equations with real coefficients

$$(\boldsymbol{\sigma}\cdot\mathbf{p})\boldsymbol{\chi}+(\mu+V-\mathcal{E})\boldsymbol{\varphi}=0,$$

$$(\mu-V+\mathcal{E})\boldsymbol{\chi}-(\boldsymbol{\sigma}\cdot\mathbf{p})\boldsymbol{\varphi}=0. \tag{4.3}$$

States with a well-defined value of the radial momentum of the membrane will be described by the system (4.3), and this system has nonvanishing solutions if the determinant of its coefficients vanishes. Hence, from the determinant of the coefficients of the system (4.3) we get

$$\mathcal{E}^2-2V\mathcal{E}+(V^2-p^2-\mu^2)=0, \tag{4.4}$$

and, solving for $\mathcal{E}$, we obtain

$$\mathcal{E}_1 = +\sqrt{p^2+\mu^2}+V, \quad \mathcal{E}_2 = -\sqrt{p^2+\mu^2}+V. \tag{4.5}$$

These results point out the consistence of the Hamiltonian operator given in Eq. (3.9), because this latter leads to energy values in agreement with the Hamiltonian (2.10). We then conclude that both solutions $\mathcal{E}_1$ and $\mathcal{E}_2$ given above yield two different forms for the time dependence of the wave function: $\mathbf{\Psi}(\mathbf{r},t)=\mathbf{\Psi}(\mathbf{r})\exp[-\mathcal{E}_1 t/\hbar]$ and $\mathbf{\Psi}(\mathbf{r},t)=\mathbf{\Psi}(\mathbf{r})\exp[-\mathcal{E}_2 t/\hbar]$. Otherwise, in analogy to the case of a particle in Dirac's theory, we can denote $\mathcal{E}_1$ as the ''positive solution'' and $\mathcal{E}_2$ as the ''negative solution'' of Dirac's equation. Finally, a relation between the components $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ can be obtained from Eq. (4.3):

$$\boldsymbol{\chi} = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{\mu+\mathcal{E}-V}\boldsymbol{\varphi}. \tag{4.6}$$

In the nonrelativistic approximation ($p \ll \mu$) we get for ''positive solutions'' $\mathcal{E}_1 - V \approx \mu$ and then Eq. (4.6) yields

$$\boldsymbol{\chi} \approx \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{2\mu}\boldsymbol{\varphi} \ll \boldsymbol{\varphi}. \tag{4.7}$$

Thus, if the membrane oscillates with a velocity small compared to the velocity of light, it follows from Eq. (4.7) that $\boldsymbol{\varphi}$ can be interpreted as the ''large component'' and $\boldsymbol{\chi}$ as the ''small component'' of the wave function, for states of ''positive solutions.'' On the other hand, for ''negative solutions'' we have $\boldsymbol{\chi} \gg \boldsymbol{\varphi}$, and then $\boldsymbol{\varphi}$ becomes the ''small component'' and $\boldsymbol{\chi}$ the ''large component''of the wave function.

V. THE MEMBRANE SPIN

Let us assume that the membrane is spinning around the Z axis, due to the action of a constant external magnetic field B_z, and that it is instantaneously without translational motion. Since the membrane has at any instant a spherical configuration then, owing to the influence of B_z, the resulting linear momentum $\mathbf{P}$ of an arbitrary point of its surface has a direction different from that one of the position vector of this point. However, we should expect that the angular momentum of rotation of the membrane $\mathbf{\Omega}=\mathbf{r}\times\mathbf{P}$ might be a constant of the motion. In order to investigate this point we calculate its time rate of change in the Heisenberg picture in Cartesian coordinates and assume the kinetic term of the Hamiltonian given in the general form (3.4). Before that we notice that $\mathbf{P}$ and $\mathbf{\Omega}$ satisfy the following commutation rules:

$$[x^j,\ P_k]=i\hbar\,\delta^j_k, \quad [P_j,\ P_k]=0, \quad [\Omega_j,\ P_k]=i\hbar\,\varepsilon_{jkl}p_l. \tag{5.1}$$

Hence, for the Ω_z component we obtain

$$\begin{aligned}\mathbf{I}[\Omega_z,H]\Phi &= \mathbf{I}([\Omega_z,P_x]\boldsymbol{\alpha}_x+[\Omega_z,P_y]\boldsymbol{\alpha}_y)\Phi \\ &= \mathbf{I}[i\hbar(\boldsymbol{\alpha}_x p_y-\boldsymbol{\alpha}_y p_x)]\Phi,\end{aligned} \tag{5.2}$$

where $\Phi(x,y,z)$ is an arbitrary function and since Ω_z commutes with f_x, f_y, $\mu(r)$, and $V(r)$, because these latter are spherically symmetric functions. Otherwise, if we evaluate the time rate of change of $\mathbf{\Sigma}_z$ when operating on $\Phi(x,y,z)$ and consider the rules (3.3)(d), (e), we get

$$\begin{aligned}\mathbf{I}[\mathbf{\Sigma}_z,H]\Phi &= \mathbf{I}([\mathbf{\Sigma}_z,\ \boldsymbol{\alpha}_x]P_x+[\mathbf{\Sigma}_z,\ \boldsymbol{\alpha}_y]P_y)\Phi \\ &\quad + i([\mathbf{\Sigma}_z,\ \boldsymbol{\alpha}_x]q_x+[\mathbf{\Sigma}_z,\boldsymbol{\alpha}_y]q_y)\kappa\Phi\boldsymbol{\beta} \\ &= \mathbf{I}[2i(\boldsymbol{\alpha}_y p_x-\boldsymbol{\alpha}_x p_y)]\Phi+\frac{2\hbar}{r^2}(x\boldsymbol{\alpha}_y-y\boldsymbol{\alpha}_x)\Phi\mathbf{I} \\ &\quad -\frac{2\hbar}{r^2}(x\boldsymbol{\alpha}_y-y\boldsymbol{\alpha}_x)\kappa\Phi\boldsymbol{\beta},\end{aligned} \tag{5.3}$$

since $q_x=\hbar x/r^2$ and $q_y=\hbar y/r^2$ and also assuming that the operator K is diagonal. Once $\Phi(x,y,z)$ is arbitrary, we thus conclude from Eqs. (5.2) and (5.3) that either for $\kappa=+1$ or for $\kappa=-1$ we can define the quantity

$$J_z=\Omega_z+\frac{1}{2}\hbar\Sigma_z \tag{5.4}$$

in such a way that $dJ_z/dt=0$, because it commutes with the Hamiltonian operator and so it is a constant of the motion. It is apparent that the same holds for the components J_x and J_y, and therefore we have a quantity $\mathbf{J}=\mathbf{\Omega}+(\hbar/2)\mathbf{\Sigma}=\mathbf{\Omega}+\mathbf{S}$, which can be taken to be the total angular momentum of the membrane, where $\mathbf{S}$ is its spin angular momentum operator.

Finally, we might also expect that states of radial oscillations of the uncharged membrane with well-defined values of momenta, should differ not only in the values $\mathcal{E}_1/p$ and $\mathcal{E}_2/p$ from Eq. (4.5), but also in the value connected to the membrane spin. In order to study this point, let us consider the free Hamiltonian and let us introduce the operator

$$S_p=\frac{\hbar}{2}(\mathbf{\Sigma}\cdot\mathbf{p}), \tag{5.5}$$

where $\mathbf{\Sigma}$ is given in Eq. (3.1). However, the commutator of the operator (5.5) with the free Hamiltonian is

$$[H,\ S_p]=\frac{\hbar}{2}[\boldsymbol{\alpha}\cdot\mathbf{p},\ \mathbf{\Sigma}\cdot\mathbf{p}], \tag{5.6}$$

and then we conclude that the operator S_p of Eq. (5.5) indeed commutes with the Hamiltonian of the uncharged membrane. Therefore, the physical quantity corresponding to the operator S_p is an integral of the motion. Once the momentum $\mathbf{p}$ is also an integral of the motion, as can be seen by direct calculation, then the physical quantity corresponding to the operator

$$S_z=\frac{\hbar}{2}\mathbf{\Sigma}_z=\frac{\hbar}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \tag{5.7}$$

is also an integral of the motion, if we take the momentum along the Z axis. Since the eigenvalues of an operator that is given by a diagonal matrix are the same values of its diagonal elements, then the eigenvalues of S_z are $\pm\hbar/2$. In an analogous way we can extend this result to the components S_x and S_y.

Let us end this section with a physical interpretation for the operator K that appears in the Hamiltonian (3.9). To start, we take the definition of $\boldsymbol{\alpha}$ given in Eq. (3.1) and write

$$\boldsymbol{\alpha}\cdot\mathbf{P}=\begin{pmatrix}\mathbf{0} & \boldsymbol{\sigma}\cdot\mathbf{P}\\ \boldsymbol{\sigma}\cdot\mathbf{P} & \mathbf{0}\end{pmatrix}. \tag{5.8}$$

Now if we consider the operator identity

$$(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B})=\mathbf{A}\cdot\mathbf{B}+i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B}) \tag{5.9}$$

that holds for the operators $\mathbf{A}$ and $\mathbf{B}$ commuting with $\boldsymbol{\sigma}$, but not necessarily with each other, we get $(\boldsymbol{\sigma}\cdot\mathbf{r})(\boldsymbol{\sigma}\cdot\boldsymbol{\Omega})=(\boldsymbol{\sigma}\cdot\mathbf{r})[\boldsymbol{\sigma}\cdot(\mathbf{r}\times\mathbf{P})]=i[(\boldsymbol{\sigma}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{P})-r^2(\boldsymbol{\sigma}\cdot\mathbf{P})]$ which leads to

$$\boldsymbol{\sigma}\cdot\mathbf{P}=\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r^2}[\mathbf{r}\cdot\mathbf{P}+i\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}]. \tag{5.10}$$

Therefore, from Eqs. (5.8) and (5.10) we find

$$\boldsymbol{\alpha}\cdot\mathbf{P}=\boldsymbol{\alpha}_r\left[P_r+i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}}{r}\right], \tag{5.11}$$

where $\boldsymbol{\alpha}_r=\boldsymbol{\alpha}\cdot\mathbf{r}/r$ is a Hermitian matrix and $P_r=(1/r)(\mathbf{r}\cdot\mathbf{p}-i\hbar)=-i\hbar\,\partial/\partial r-i\hbar/r$. Otherwise, the scalar product $\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}$ that appears in Eq. (4.1) means

$$\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}=\boldsymbol{\alpha}\cdot\mathbf{P}+i\frac{\hbar}{r}\boldsymbol{\alpha}_r\boldsymbol{\beta}K, \tag{5.12}$$

and thus if this operator acts on an arbitrary function $\Phi(r,\theta,\phi)$ we have, by considering Eq. (5.11),

$$(\boldsymbol{\alpha}\cdot\boldsymbol{\Pi})\Phi=\boldsymbol{\alpha}_r\left[P_r+i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}}{r}+i\frac{\hbar}{r}\boldsymbol{\beta}K\right]\Phi. \tag{5.13}$$

From this matrix equation we conclude that either for $\kappa=+1$ or $\kappa=-1$ we obtain

$$\boldsymbol{\alpha}\cdot\boldsymbol{\Pi}=\boldsymbol{\alpha}_r\left[P_r+i\frac{\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}+\hbar}{r}\right]. \tag{5.14}$$

If we define the operator K in a similar way as in the relativistic electron theory by means of

$$\hbar K=\boldsymbol{\beta}[\boldsymbol{\sigma}\cdot\boldsymbol{\Omega}+\hbar], \tag{5.15}$$

we conclude that both the Hamiltonian (3.9) and its equivalent form (4.1) contain naturally the above operator, and that K is connected to the total angular momentum of the membrane. Moreover, since K commutes with $\boldsymbol{\alpha}_r$, $\boldsymbol{\beta}$, and P_r, then it commutes as well with the complete Hamiltonian (3.9), and hence it is a constant of the motion. Otherwise, using the operator identity $\boldsymbol{\Omega}\times\boldsymbol{\Omega}=i\hbar\boldsymbol{\Omega}$ we are led to

$$\hbar^2K^2=\left(\boldsymbol{\Omega}+\frac{1}{2}\hbar\boldsymbol{\sigma}\right)^2+\frac{1}{4}\hbar^2=\mathbf{J}^2+\frac{1}{4}\hbar^2, \tag{5.16}$$

where $\mathbf{J}$ is the total angular momentum operator defined below Eq. (5.4). It is easy to verify that the operator $\hbar^2K^2$ is an integral of the motion and that it has the eigenvalues $\hbar^2\kappa^2$, where $\kappa=\pm(j+1/2)$ and j are the eigenvalues of $\mathbf{J}$. However, we remind the reader that the only possible values of κ that allow the linearization performed in Sec. III are ±1, and also, that the spin $\pm\hbar/2$ of the membrane only holds for $\kappa=\pm1$. Thus, according to the present approach, the unique value allowed for j should be 1/2.

VI. THE CASE OF THE UNCHARGED MEMBRANE

At the limiting case when the membrane charge tends to zero we obtain from Eq. (4.3)

$$\boldsymbol{\varphi}=(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\chi},\quad \boldsymbol{\chi}=(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\varphi}, \tag{6.1}$$

where $\mathbf{n}=\mathbf{p}/\mathcal{E}$ is an unit vector. The vector $\mathbf{n}$ is parallel to the momentum of the membrane for positive solutions, and it is antiparallel to that momentum for negative solutions. Thus, the spatial part $\boldsymbol{\Psi}(\mathbf{r})$ of the wave function becomes

$$\boldsymbol{\Psi}(\mathbf{r})=\begin{pmatrix}\boldsymbol{\varphi}\\ (\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\varphi}\end{pmatrix}. \tag{6.2}$$

Otherwise, when we act upon the wave function $\boldsymbol{\Psi}(\mathbf{r})$ with the pseudoscalar $\boldsymbol{\sigma}\cdot\mathbf{n}$, their components change places owing to the results given in Eq. (6.1)

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\Psi}(\mathbf{r})=\begin{pmatrix}\boldsymbol{\chi}\\ \boldsymbol{\varphi}\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}\begin{pmatrix}\boldsymbol{\varphi}\\ \boldsymbol{\chi}\end{pmatrix} \tag{6.3}$$

and we conclude that the action of the operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ upon the spatial part of the wave function of an uncharged membrane is equivalent to the action of the matrix $-i\boldsymbol{\gamma}_5$,

$$(\boldsymbol{\sigma}\cdot\mathbf{n})=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{I} & \mathbf{0}\end{pmatrix}=-i\boldsymbol{\gamma}_5, \tag{6.4}$$

where $\boldsymbol{\gamma}_5=\boldsymbol{\gamma}_3\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0$, $\gamma_0=\beta$, and $\gamma_i=\beta\alpha_i$, $i=1,2,3$. Now if we introduce two linear combinations of the functions $\boldsymbol{\varphi}$ and $\boldsymbol{\chi}$ by means of

$$\boldsymbol{\Phi}=\frac{1}{2}(\boldsymbol{\varphi}+\boldsymbol{\chi})=\frac{1}{2}[\mathbf{I}+(\boldsymbol{\sigma}\cdot\mathbf{n})]\boldsymbol{\varphi},$$

$$\boldsymbol{F}=\frac{1}{2}(\boldsymbol{\varphi}-\boldsymbol{\chi})=\frac{1}{2}[\mathbf{I}-(\boldsymbol{\sigma}\cdot\mathbf{n})]\boldsymbol{\varphi}, \tag{6.5}$$

we easily see that the functions $\boldsymbol{\Phi}$ and $\mathbf{F}$ satisfy the respective relations

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\boldsymbol{\Phi}=\boldsymbol{\Phi},\quad (\boldsymbol{\sigma}\cdot\mathbf{n})\mathbf{F}=-\mathbf{F} \tag{6.6}$$

which allow us to conclude that the two-component functions $\boldsymbol{\Phi}$ and $\mathbf{F}$ are the eigenfunctions of the operator $(\boldsymbol{\sigma}\cdot\mathbf{n})$, this latter defining the component of the membrane spin along the direction of its momentum. Then, according to Eqs. (6.6) the operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ has the eigenvalues $+1$ and -1, which define the helicity of the membrane. Otherwise, since the actions of the operators $(\boldsymbol{\sigma}\cdot\mathbf{n})$ and $-i\boldsymbol{\gamma}_5$ are equivalent, according to Eq. (6.4), then we see from Eq. (6.5) that their respective eigenfunctions are

$$\boldsymbol{\Phi}=\frac{1}{2}(\mathbf{I}-i\boldsymbol{\gamma}_5)\boldsymbol{\Psi},\quad \mathbf{F}=\frac{1}{2}(\mathbf{I}+i\boldsymbol{\gamma}_5)\boldsymbol{\Psi}. \tag{6.7}$$

This means that when we multiply the four-component wave function $\mathbf{\Psi}(\mathbf{r})$ by $(\mathbf{I}\pm i\boldsymbol{\gamma}_5)$, it changes into a two-component function. We then conclude that each value of the momentum corresponds to a state with a well-defined helicity and to only one spin state. For positive helicity the momentum and the spin are parallel for states where $\mathcal{E}=+p$, and if the helicity is negative they are antiparallel for states where $\mathcal{E}=-p$. Such states can be only realized for uncharged membranes, which should move with the velocity of light. Hence, a longitudinal polarization of the uncharged membrane, in the sense of the direction of its spin, is uniquely connected to the direction of its momentum. That is only possible if the membrane rest energy is null.

VII. CONCLUDING REMARKS

The present approach leads to extended neutral and massless particles with spin $\pm\hbar/2$. The Dirac equation postulates in this case positive and negative states, which can be considered as ''charge conjugate'' states. Once the term ''charge conjugate'' is meaningless for uncharged membranes, it is better to speak about states corresponding to a massless lepton and to its corresponding antiparticle. Thus, if we accept this model for neutrinolike particles, and since it is already well established that the neutrino and the antineutrino are different particles, then they differ in helicity according to the present framework, as should be expected. The next step is to determine the energy levels of the charged membrane by means of Dirac's equation. This is a subject under investigation.

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